THE JACOBIAN DETERMINANT OF THE CONDUCTIVITIES-TO-RESPONSE-MATRIX MAP FOR WELL-CONNECTED CRITICAL CIRCULAR PLANAR GRAPHS

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ABSTRACT. We consider the map from conductivities to the response matrix. For critical circular planar graphs, this map is known to be invertible, at least when the conductivities are positive. We calculate the Jacobian determinant of this map, which turns out to have a fairly simple form. Using this we show that for arbitrary critical circular planar networks, the map from conductivities to the response matrix is generally invertible when the conductivities are allowed to be negative or complex (but nonzero). This is an alternate proof to a result in [4].

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1. INTRODUCTION

Suppose we have a critical circular planar graph Γ. Then it is known that given a set of positive conductivities γ on the edges of Γ, the values of γ can be determined from the response matrix Λ. That is, the conductivities are recoverable from the response matrix. Refer to [2] for the proof of this and basic theory of electrical network recovery. If there are n boundary nodes, then Λ has $\binom{n}{2}$ degrees of freedom, corresponding to the edges in the complete graph K_n that Γ is equivalent to. If the number of edges in Γ is also $\binom{n}{2}$, then we can talk about the Jacobian determinant of the $\gamma \to \Lambda$ map. For such a graph, there are as many crossings in the medial graph as possible, so this is equivalent to Γ being well-connected.

The recovery algorithm (in [2]) for critical circular planar graphs uses only rational functions – thus the map from γ to Λ is birational. Consequently, if we let the conductivities take values over the complex numbers, then the graph will be generically recoverable. In this paper, we show that the reverse maps are singular only when some conductivities are zero, or when the response matrix does not exist. It follows that negative and complex (but nonzero) conductivities are recoverable, as long as the response matrix exists. This was previously shown by Michael Goff in [4], using completely different techniques.

2. Preliminaries

We begin by fixing some notation. For any $n \in \mathbb{N}$, 1_n will denote the vector in $\mathbb{R}^n$ whose entries are all one. The space orthogonal to 1_n will be denoted $1_n^\perp$, and the projection from $\mathbb{R}^n$ to $1_n^\perp$ will be Π_n. The space of symmetric $n \times n$ matrices will be denoted Sym_n The set of invertible elements of Sym_n will be denoted Sym_n^*. Similarly, the space of $n \times n$ matrices with row (and column) sums zero will be denoted Kir_n. The set Kir_n^* will denote the matrices in Kir_n whose kernel is exactly the span of 1_n. The spaces Kir_n and Kir_n^* can be identified with the spaces of self-adjoint and invertible self-adjoint operators on $1_n^\perp$.

For each n, we can choose some arbitrary norm-preserving isomorphism $g_n : 1_n^\perp \to \mathbb{R}^{n-1}$. This gives rise naturally to an isomorphism h_n between the spaces Kir_n and Sym_{n-1}. This map also provides an isomorphism between Kir_n^* and Sym_{n-1}^*.

Lemma 2.1. *If $X \in \mathrm{Kir}_n$, then $\det(h_n(X))$ equals n times the determinant of any $(n-1) \times (n-1)$ principal submatrix of X.*

Proof. Since S is symmetric, it is diagonalizable. So is $h_n(X)$. The determinant of $h_n(X)$ will just be the product of the eigenvalues (counting multiplicities) of $h_n(X)$. Now the eigenvalues of X are the same as the eigenvalues of $h_n(X)$, with the addition of the eigenvalue 0. Therefore, $\det(h_n(X))$, the product of the eigenvalues of $h_n(X)$, will just be

$$\frac{\partial}{\partial z} \prod_{\lambda} (z + \lambda)$$

evaluated at $z = 0$, where λ ranges over the eigenvalues of X. This is just

$$\frac{\partial}{\partial z} \det(X + z)$$

evaluated at $z = 0$. Using the cofactor expansion, this is just

$$\sum_{i=1}^{n} X_{(ii)}$$

where $X_{(ii)}$ is the determinant of the $(n-1) \times (n-1)$ submatrix obtained by deleting row i and column i. Since X has row and column sums 0, the $X_{(ii)}$ are all equal. Thus $\det(h_n(X))$ is just $n \det(X_{(ii)})$. □

Let Γ be a well-connected critical circular planar graph whose vertices V are partitioned into $\operatorname{int} V$ and ∂V. Γ will be fixed for most of the discussion. Let γ be a vector of conductivities on the edges of Γ. Partition the Kirchhoff matrix K (whose off-diagonal entries are all of the form $-\gamma_{ij}$) in the usual way as

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

where A is the matrix of conductivities between ∂V and ∂V, C is the matrix of conductivities between $\operatorname{int} V$ and $\operatorname{int} V$, and B is the matrix of conductivities between ∂V and $\operatorname{int} V$.

Let $\mathcal{S}$ be the set of spanning trees of Γ, considered as sets of edges. Let $\mathcal{T}$ be the collections of all sets of edges which are acyclic, connect every interior node to the boundary, and do not connect any boundary nodes nodes. Equivalently, $\mathcal{T}$ contains all the sets that are spanning trees of the graph obtained by identifying all the boundary nodes of Γ. The following two polynomials in the conductivities will play significant roles in what follows:

$$\Sigma = \sum_{X \in \mathcal{T}} \prod_{e \in X} \gamma(e)$$

$$\Upsilon = \sum_{X \in \mathcal{S}} \prod_{e \in X} \gamma(e)$$

By the determinant tree formula of [5], $\Sigma = \det(C)$. Likewise, Υ is the determinant of any $|V| - 1 \times |V| - 1$ principal submatrix of K. (All of these determinants will be equal because the row and column sums of K vanish).

The response matrix Λ is the Schur Complement

$$\Lambda = A - BC^{-1}B^T. \tag{1}$$

This matrix is characterized by the fact that for a function u on the boundary vertices ∂V,

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \cdot \begin{pmatrix} u \\ \phi \end{pmatrix} = \begin{pmatrix} \Lambda u \\ 0 \end{pmatrix}$$

for some function ϕ on the interior.

Since $\Lambda \in \mathrm{Kir}_n$, it has $\binom{n}{2}$ degrees of freedom, where n is the number of boundary nodes. In particular, Λ carries the same information as Λ_{ij}, for $1 \leq i < j \leq n$. Let J be the Jacobian determinant of the map from γ to the vector of Λ_{ij} $(i < j)$. J will depend on the ordering of variables, which we do not fix, so J will only be defined up to a sign change.

Theorem 2.2. *J is a rational function in the conductivities. Its denominator is a factor of a power of Σ.*

Proof. From Equation 1, the entries in Λ are given by rational functions, each of which has at most $\det(C) = \Sigma$ in the denominator. Consequently, the derivatives in the Jacobian matrix will all be rational functions with Σ^2 in the denominator. The determinant of the Jacobian matrix will clearly have the desired property, then. □

Analogous to Λ, there is a Neumann-to-Dirichlet map H, studied in [1]. The map H is characterized by the fact that if $x, y \in 1_n^\perp$, then

$$x = \Lambda y \iff y = Hx.$$

Consequently, H amounts to a pseudoinverse of Λ. In terms of the h_n maps defined at the start of this section, $H = h_n^{-1}((h_n(\Lambda))^{-1})$

If we identify H with the vector of H_{ij} for $i < j$, then we can analogously construct the Jacobian determinant

$$\overline{J} = \det\left(\frac{\partial H}{\partial \gamma}\right).$$

Just as the denominator of J is a power of Σ, the denominator of $\overline{J}$ will turn out to be a power of Υ.

K itself has a pseudoinverse of the same sort, K^*. This is merely the Neumann-to-Dirichlet map of the graph obtained by making all nodes of Γ boundary nodes. K^* has the property that for $x, y \in 1_{|V|}^{\perp}$,

$$x = Ky \iff y = K^* x \tag{2}$$

Lemma 2.3. *The entries in K^* are rational functions in the conductivities, with Υ in their denominators.*

Proof. Let $m = |V|$. By the defining property of K^*, Equation 2, we must have $h_m(K^*)$ and $h_m(K)$ inverses of eachother. By Lemma 2.1, $\det(h_m(K)) = mK_{(ii)}$, where $K_{(ii)}$ is the determinant of an $(m-1) \times (m-1)$ principal submatrix of K. By the determinant-tree formula, $K_{(ii)} = \Upsilon$. Therefore, $\det(h_m(K)) = m\Upsilon$.

It follows that the entries in $h_m(K)^{-1}$ are all rational functions in the conductivities, with denominators Υ. Then so are all the entries in $h_m^{-1}((h_m(K))^{-1}) = K^*$, because h_m^{-1} is a linear function, □

Theorem 2.4. *$\overline{J}$ is a rational function whose denominator is a (factor of a) power of Υ.*

Proof. Suppose we know K^*. That is, we know how to determine the voltages of the graph from the total current flowing out of each node. Given a set of boundary currents (adding to zero), if we extend them to equal zero on the interior, and apply $K*$, and then throw away the interior voltages, we get the Dirichlet data corresponding to the original Neumann data. Therefore, H can be expressed in terms of K^*. Specifically, for $x \in \mathbb{R}^n$

$$Hx = \Pi_n \begin{pmatrix} I_n & 0 \end{pmatrix} K^* \begin{pmatrix} I_n \\ 0 \end{pmatrix} \Pi_n x.$$

Then, since H depends linearly on K^*, the entries in H are rational functions in the conductivities, all with denominator Υ. As in the case for J, the entries in the matrix of partial derivatives will all have some factor of Υ^2 in their denominators, and so $\overline{J}$ will have a power of Υ in its denominator. □

Our approach for calculating J proceed in two parts: we will first show that the denominator of J is at most a factor of an appropriate power of Σ, utilizing the relationship between $\overline{J}$ and J. Then we will show that the numerator of Σ is at least a multiple of a certain polynomial in the conductivities. Since the numerator and denominator of J must be homogeneous polynomials of the same degree, it will follow that J is determined, up to a scalar multiple. Actually, we will only show this for a specific graph (the Towers of Hanoi graph), and then show that the formula for the Jacobian transforms in the correct way under $Y-\Delta$ transformations, which will suffice because all well-connected critical graphs on n boundary nodes are $Y-\Delta$-equivalent (by Theorem 8.7 of [2]).

3. The Jacobian of Inversion

In this section we calculate the Jacobian of the map which sends Λ to H.

Lemma 3.1. *Let $f : \mathrm{Sym}_m^* \to \mathrm{Sym}_m^*$ send a matrix X to X^{-1}. Then the Jacobian determinant of f is given as follows:*

$$\det\left(\frac{\partial f(X)}{\partial X}\right) = \beta(\det(X)^{-1-m}),$$

for some scalar β.

Proof. Suppose first that X is a diagonal matrix, so that $X_{ii} = \chi_i$ for some numbers χ_i, and $X_{ij} = 0$ if $i \neq j$. In this case, it will turn out that varying $X_{ij} = X_{ji}$ slightly will only effect $(X^{-1})_{ij}$, that is,

$$\frac{\partial f(X)_{ij}}{\partial X_{kl}} = 0$$

unless $\{i, j\} = \{k, l\}$. For example, in the 3×3 case, if we vary the entry X_{11}, then

$$\begin{pmatrix} \chi_1 + \epsilon & 0 & 0 \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix}^{-1} = \begin{pmatrix} (\chi_1 + \epsilon)^{-1} & 0 & 0 \\ 0 & \chi_2^{-1} & 0 \\ 0 & 0 & \chi_3^{-1} \end{pmatrix}$$

so the only term in the inverse which varies is $(X^{-1})_{11}$, and the derivative is given as

$$\frac{\partial (X^{-1})_{11}}{\partial X_{11}} = \frac{-1}{\chi_1^2}.$$

Likewise, if we vary X_{12} and X_{21}, then

$$\begin{pmatrix} \chi_1 & \epsilon & 0 \\ \epsilon & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\chi_2}{\chi_1\chi_2 - \epsilon^2} & \frac{-\epsilon}{\chi_1\chi_2 - \epsilon^2} & 0 \\ \frac{-\epsilon}{\chi_1\chi_2 - \epsilon^2} & \frac{\chi_1}{\chi_1\chi_2 - \epsilon^2} & 0 \\ 0 & 0 & \chi_3^{-1} \end{pmatrix}$$

so the only term with a nonzero derivative is

$$\frac{\partial (X^{-1})_{12}}{\partial X_{12}} = \frac{\partial (X^{-1})_{21}}{\partial X_{12}} = \frac{-1}{\chi_1\chi_2}.$$

In general, we have

$$\frac{\partial (X^{-1})_{ij}}{\partial X_{kl}} = 0$$

unless $\{i, j\} = \{k, l\}$, in which case

$$\frac{\partial (X^{-1})_{ij}}{\partial X_{ij}} = \frac{\pm 1}{\chi_i \cdot \chi_j}.$$

Therefore, the Jacobian matrix of f at X is *diagonal*, and so the Jacobian determinant at X is

$$\prod_{1 \le i \le j \le m} \frac{\pm 1}{\chi_i \cdot \chi_j} = \pm \prod_{1 \le i \le m} \chi_i^{-m-1} = \pm \det(X)^{-m-1}.$$

Now, suppose X is not diagonal. Since X is symmetric, there is some orthogonal matrix O such that OXO^{-1} is diagonal. Let g_O be the map on Sym_m^* which sends $Y \to OYO^{-1}$. It is easily seen that $g_O^{-1} \circ f \circ g_O = f$. By the chain rule, then,

$$\det(\partial g_O^{-1}(f(g_O(X)))) \det(\partial f(g_O(X))) \det(\partial g_O(X)) = \det(\partial f(X)).$$

Now $\det(\partial g_O(Y))$ and $\det(\partial g_O^{-1}(Y))$ are constants (not depending on Y), since g_O is linear. Therefore they are inverses of each other, and they cancel out. So

$$\det(\partial f(g_O(X))) = \det(\partial f(X)).$$

But

$$\det(\partial f(g_O(X))) = \det(g_O(X))^{-m-1} = \pm \det(OXO^{-1})^{-m-1} = \pm \det(X)^{-m-1},$$

since $g_O(x)$ is diagonal. $\square$

Theorem 3.2. *The Jacobian determinant of the map which sends Λ to H is $\pm\alpha(\Sigma/\Upsilon)^n$ for some constant α.*

Proof. Λ and H take values in Kir_n^*, and the map which sends $\Lambda \to H$ is simply $H = h_n^{-1}((h_n(\Lambda))^{-1})$. Therefore, by the chain rule, the fact that h_n is linear, and Lemma 3.1, the Jacobian determinant of the map which sends Λ to H is $\pm \det(h(\Lambda))^{-n}$. By Lemma 2.1, $\det(h(\Lambda))$ is just n times the determinant of an $(n-1) \times (n-1)$ principal submatrix of Λ. By the Determinant-Tree Formula, the determinant of any $(n-1) \times (n-1)$ principal submatrix of Λ is just Υ/Σ. Thus

$$\det\left(\frac{\partial H}{\partial \Lambda}\right) = \pm \det(h(\Lambda))^{-n} = \frac{\pm\Sigma^n}{(n\Upsilon)^n}.$$

□

Theorem 3.3. *The denominator of J is (a factor of) $\Sigma^n\Upsilon^N$ for some N.*

Proof. From the chain rule and the previous theorem, we know that

$$J = \frac{\alpha\overline{J}\Upsilon^n}{\Sigma^n}$$

for some constant α. Also, from Theorem 2.4, we know that $\overline{J}$ is a rational function whose denominator is a (factor of a) power of Υ. The desired result now follows. □

Lemma 3.4. *If Υ is factored as $\Upsilon = \Upsilon_1\Upsilon_2$, then no variable occurs in both Υ_1 and Υ_2. Also, if C is a simple cycle in Γ, then for edges $e_1, e_2 \in C$, the variables γ_{e_1} and γ_{e_2} are in the same factor of Υ.*

Proof. Since Υ is linear in each variable, no variable could occur in both Υ_1 and Υ_2. Therefore, the variables in Υ_1 and Υ_2 are completely disjoint. Every conductivity occurs in Υ, because any edge can be extended to make a minimal spanning tree, since there are no self-loops in Γ. Thus the factorization into $\Upsilon_1 \cdot \Upsilon_2$ partitions the edges of Γ into two sets, E_1 and E_2. Also, when we multiply Υ_1 and Υ_2, no terms cancel out – each term of Υ comes from a unique monomial in Υ_1 and a unique monomial in Υ_2. Define the sets

$$\mathcal{T}_1 = \{T \subseteq E_1 \ : \ \left(\prod_{e\in T}\gamma_e\right) \text{ is a monomial in } \Upsilon_1\}$$

and

$$\mathcal{T}_2 = \{T \subseteq E_2 \ : \ \left(\prod_{e\in T}\gamma_e\right) \text{ is a monomial in } \Upsilon_2\}.$$

Then a set of edges T is a tree ($T \in \mathcal{T}$) iff $T \cap E_1 \in \mathcal{T}_1$ and $T \cap E_2 \in \mathcal{T}_2$, because of the factorization of Υ.

Now suppose C is a cycle which intersects both E_1 and E_2. Let $C_i = C \cap E_i$. Then since C is a minimal cyclic set, both C_1 and C_2 are acyclic. Therefore, C_1 and C_2 can be extended to minimal spanning trees of Γ, T_1 and T_2 respectively. Then $T_1 \in \mathcal{T} \Rightarrow T_1 \cap E_1 \in \mathcal{T}_1$ and likewise $T_2 \cap E_2 \in \mathcal{T}_2$. Thus $(T_1 \cap E_1) \cup (T_2 \cap E_2) \in \mathcal{T}$, by the above comments. However,

$$C = (C_1 \cap E_1) \cup (C_2 \cap E_2) \subseteq (T_1 \cap E_1) \cup (T_2 \cap E_2),$$

so $(T_1 \cap E_1) \cup (T_2 \cap E_2)$ cannot be a tree, and we have a contradiction. □

At this point, we restrict to the case where Γ is one of the Towers of Hanoi graphs, shown in Figure 1.

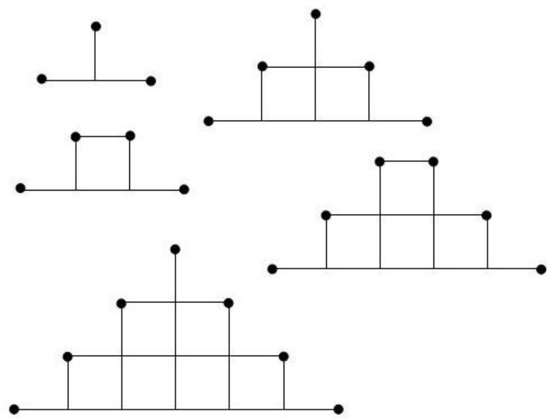

FIGURE 1. The well-connected Tower of Hanoi graphs, with 3 to 7 boundary nodes. Filled in circles are boundary nodes, other crossings are interior nodes. Note the plethora of cycles of length 4.

Lemma 3.5. *If Γ is the Towers of Hanoi graph with $n \geq 3$ boundary nodes, then Σ and Υ have no common factors.*

Proof. The case where $n = 3$ is easily verified, since $\Upsilon = \gamma_1\gamma_2\gamma_3$ and $\Sigma = \gamma_1+\gamma_2+\gamma_3$. So suppose $n > 3$.

If e is a boundary spike of Γ, then every spanning tree of Γ necessarily contains e, so $\gamma_e|\Upsilon$. The Towers of Hanoi graph always has two or three boundary spikes. Let Υ_0 be the factor leftover after dividing by the conductivity of the boundary spikes. All the remaining conductivities occur as variables in Υ_0. If we could factor Υ_0 as a product $\Upsilon_1 \cdot \Upsilon_2$, we would have partitioned the remaining variables up into two sets, in such a way that no simple cycle is divided. This is clearly impossible because every 1×1 square in the Towers of Hanoi graph is a simple cycle, so Υ_0 is prime.

Now Σ is not divisible by any monomials, because no edge occurs in all tree diagrams. (This is equivalent to saying that every edge occurs in some minimal spanning tree of the dual graph, which is valid because the dual graph has no self-loops). Therefore, if any factor of Υ is also a factor of Σ, it is Υ_0.

If V is the number of vertices in the graph, then the degree of Υ is $V - 1$, since a tree has one less edge than vertex. Likewise, the degree of Σ is $V - n$, since a tree diagram of a graph Γ is just a spanning tree of the graph obtained by merging all the boundary nodes of Γ. Since there are at most 3 boundary spikes in Γ, the degree of Υ_0 is at least $V - 4$. Therefore, if $\Upsilon_0|\Sigma$, $V - 4 \leq V - n$, so $n \leq 4$. However, if $n = 4$, then there are only two boundary spikes, and so the degree of Υ_0 is $V - 3$, which exceeds the degree of Σ. Consequently, $\Upsilon_0 \nmid \Sigma$, and Σ has no factors in common with Υ. $\square$

Theorem 3.6. *For Γ a Towers of Hanoi Graph, J is a rational function whose denominator is (a factor of) Σ^n*

Proof. By Theorem 2.2 the denominator is a factor of Σ^M for some M. Also, by Theorem 3.3 it is of the form $\Sigma^n \Upsilon^N$ for some N. Since Σ and Υ have no common factors by Lemma 3.5, the result follows easily. □

Incidentally, most of the results of this section can be extended to work for all well-connected graphs; only Lemma 3.5 requires additional complications. The real reason for using the Towers of Hanoi is that the arguments in the next section depend heavily on the layout of the graph, and it is unclear how to generalize them for other well-connected graphs.

4. The numerator

In this section, we show that the numerator of J is a monomial(!) of a specific form. Actually, we only show this for the Towers of Hanoi graph, because it is unclear how to proceed in general. In a subsequent section, we show how the main result is invariant under $Y-\Delta$ transformations.

Suppose that we take an edge e in the well-connected graph Γ and delete it. The new graph Γ' may or may not be critical. As Jeffrey Giansiricusa previously showed in [3], a non-critical graph Γ will have some number of degrees of freedom $k(\Gamma)$ within which its conductivities can be varied, without changing the response matrix. In particular then, if we delete the edge e from a critical graph Γ and obtain a graph Γ', then $k(\Gamma')$ is just the nullity of the Jacobian matrix $\frac{\partial \Lambda}{\partial \gamma}$ when $\gamma_e = 0$ and $\gamma_{e'} > 0$ for $e' \neq e$. The nullity does not depend on the conductivities, as long as all of them but e's are positive. The value $k(\Gamma')$ can be determined by emptying lenses and uncrossing empty lenses until a critical graph is produced, and then counting the number of uncrossings needed, which will equal $k(\Gamma')$.

It is clear from the means in which lenses are emptied (as described in the proof of Lemma 8.2 of [2]) that any simple lens can be emptied, and once uncrossed, the geodesics could simply be moved back to their original location (see Figure 2). Consequently, to remove lenses we can dispense with the Y-Δ transformations and simply uncross one of the poles of any simple lens. Here we use "simple lens" to mean one which does not contain any smaller lens within it. Two simple lenses may overlap eachother, but neither can contain the other. In what follows, we will not bother with emptying lenses or moving geodesics around. Instead, we will merely smooth crossings in the medial graph.

Lemma 4.1. *If $M(x)$ is an $m \times m$ matrix whose entries are functions of some parameter x, and the functions are C^∞ for x in some neighborhood of 0, and if $M(0)$ has nullity n, then*

$$\frac{d^i \det(M(x))}{dx^i}\Big|_{x=0} = 0$$

for $0 \leq i < n$.

Proof. Proof by induction on n. If $n = 0$, there is nothing to prove. Otherwise $n \geq 1$, so $M(0)$ is singular, and $\det(M(0)) = 0$. Then we only need the first $n-1$ derivatives of $\det(M(x))$ to vanish at $x = 0$. We can think of the determinant as a multilinear product of the columns of $M(x)$. Letting $M(x)_j$ denote the jth column, then

$$\det(M(x)) = \langle M(x)_1, M(x)_2, \ldots M(x)_m\rangle$$

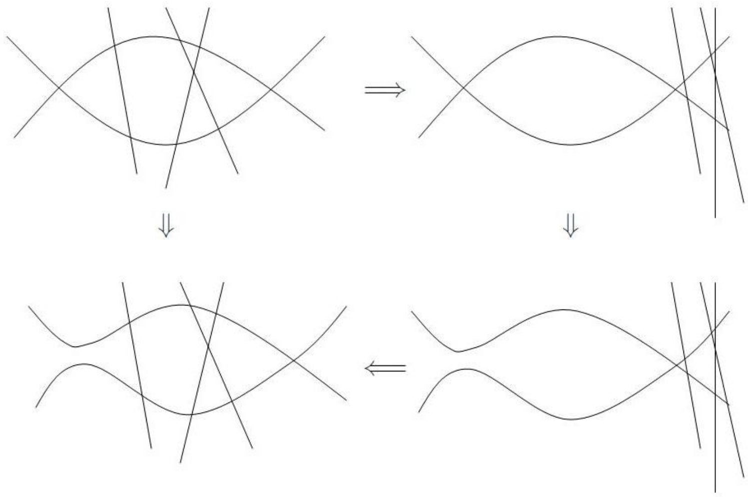

FIGURE 2. A simple lens can be emptied, uncrossed, and then filled back up. A shorter operation which accomplishes the same result is to just uncross one of the poles of the lens. This is valid because the method for emptying lenses in Lemma 8.2 of [2] only moves geodesics across one pole.

Then by the product rule,

$$\frac{d\det(M(x))}{dx} = \langle M'(x)_1, M(x)_2, \dots M(x)_m\rangle + \dots + \langle M(x)_1, M(x)_2, \dots M'(x)_m\rangle$$

Now all the terms in this sum are determinants in their own right, of matrices which are satisfy the conditions of the inductive hypothesis, but with nullities at most $n-1$. By the inductive hypothesis, then, the right hand side is a sum of functions whose values and first $n-2$ derivatives vanish at $x=0$. Therefore, the first $n-1$ derivatives of $\det(M(x))$ vanish. □

Lemma 4.2. *If the nullity of the Jacobian matrix* $\frac{\partial\Lambda}{\partial\gamma}$ *is* k *when* $\gamma_e = 0$ *and all other* $\gamma_{e'} > 0$, *then the numerator of* J *is divisible by* γ_e^k.

Proof. As noted in the proof of lemma 3.5, Σ is not divisible by any monomials. Therefore, if we let γ_e be 0, but keep all other variables positive, Σ will remain positive, and J will be C^∞ as a function of the conductivities. Then by the previous lemma,

$$\frac{\partial^i J}{\partial\gamma_e^i}\bigg|_{\gamma_e=0} = 0$$

for $0 \le i < k$. If we express J as a ratio of two polynomials, $J = N/D$, then $N = JD$, so by the general Leibniz Rule

$$\frac{\partial^i N}{\partial\gamma_e^i} = \sum_{j=0}^{i}\binom{i}{j}\frac{\partial^j J}{\partial\gamma_e^j}\frac{\partial^{i-j} D}{\partial\gamma_e^{i-j}},$$

which vanishes if $i < k$ and $\gamma_e = 0$. So N and its first $k-1$ derivatives with respect to γ_e vanish, whenever all other variables are positive. This is only possible if $\gamma_e^k | N$, since N is a polynomial. □

Definition 4.3. *If e is an edge in a well-connected graph Γ, the* exponent *of e,* ex(e) *is the number of geodesics in the medial graph which have endpoints along the boundary arc between the endpoints of two rays which shoot off from the same side of e (Figure 3).*

Which side of e we choose to shoot the rays off of makes no difference because Γ is well-connected. For an example, see Figure 4.

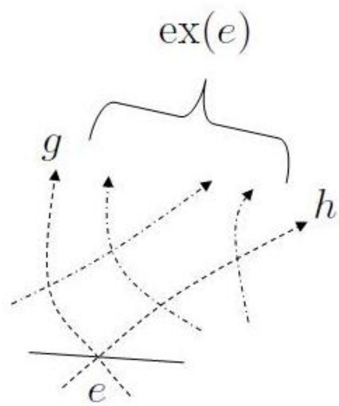

FIGURE 3. To determine the exponent ex(e) of an edge e, shoot two rays g and h off the same side of e. Then count the number of geodesics' endpoints between the ends of g and h (not counting g and h).

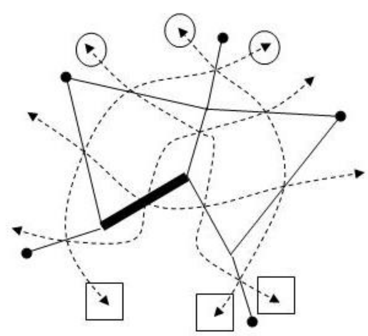

FIGURE 4. The exponent of the bold edge is 3, either because of the three circled geodesic endpoints, or the three boxed endpoints.

Theorem 4.4. *For Γ_n the Towers of Hanoi graph with n boundary nodes, if we delete an edge e, then the degeneracy of the resultant graph is equal to the exponent of the edge e.*

Proof. There are two types of edges in a Towers of Hanoi graph: horizontal and vertical edges.

If e is a vertical edge, then we draw a line down and to the left from e, smoothing all crossings (i.e., deleting all edges in the original graph) along this line until we reach the lowest level. See Figure 5. If we do these smoothings in order from top to bottom (right to left), it is easily seen that at each point, the smoothed crossing was a pole of a simple lens. (The other pole will in general be on the bottom row - drawing a picture makes this clear). Also, the final graph obtained after these smoothings in lensless. Moreover, the crossings that are smoothed in the process are exactly those crossings where a geodesic that starts between the endpoints of the two rays obtained by shooting off to the left cross one of the two geodesics through e. So the number of smoothings, which equals the degneracy, is just equal to the exponent.

A similar approach handles the case that e is a horizontal edge. Here, we delete all the horizontal edges that are directly northeast (Figure 6) and directly northwest from e (Figure 7). Again, a picture demonstrates why this works, and why the number of smoothings equals the exponent of e. □

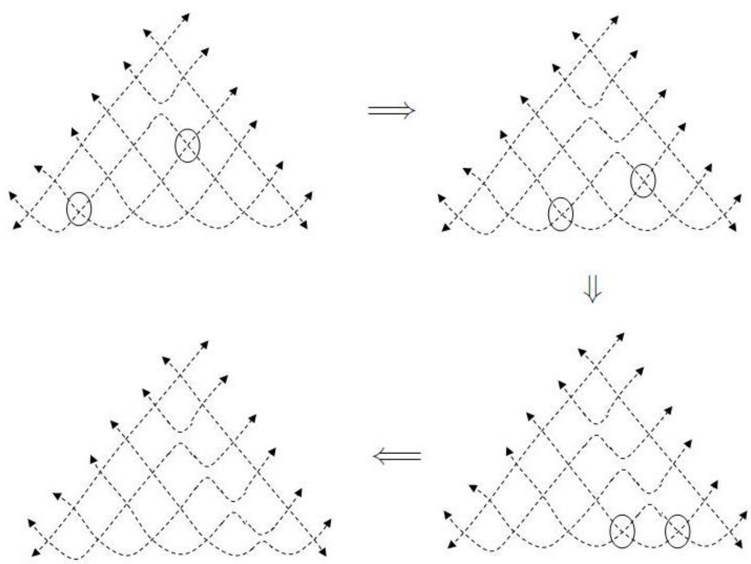

FIGURE 5. After deleting a vertical edge in the Towers of Hanoi graph, the medial graph on the top left results. At each step, we identify a simple lens and uncross one of the poles of the lens. In this case, three uncrossings are necessary, so the degneracy is 3. Three is also the exponent of the edge that was deleted.

Lemma 4.5. *If Γ is the towers of Hanoi graph with n boundary nodes, then the number of edges in a tree diagram is $\frac{1}{n}\sum_{e\in E}\mathrm{ex}(e)$.*

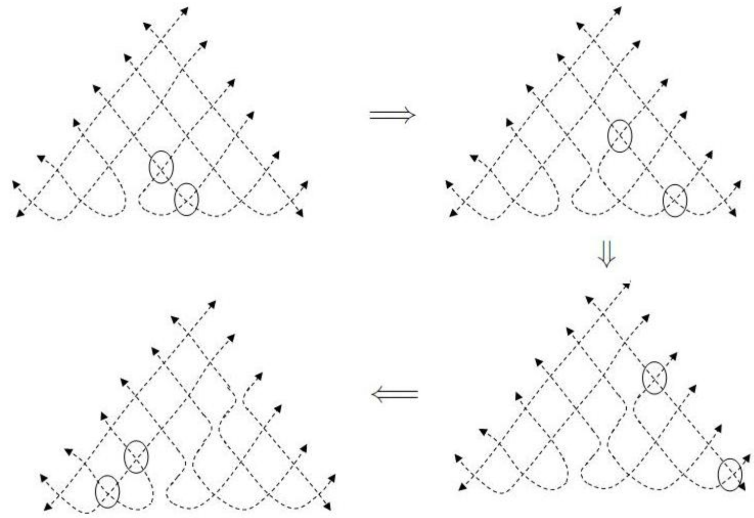

FIGURE 6. After deleting a horizontal edge on the bottom row, we start to remove lenses by smoothing crossings to the northeast of the deleted edge. See Figure 7 for the second half.

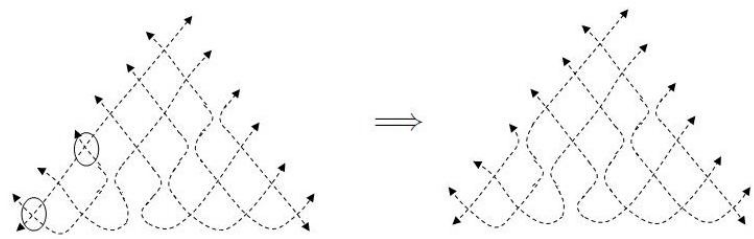

FIGURE 7. After making a series of smoothings to the northeast, we make additional smoothings to the northwest of the deleted edge. See Figure 6 for the first half. In total, five lenses were emptied, so the graph had five degrees of degeneracy. Five is also the exponent of the deleted edge.

Proof. The sum can be calculated in a straightforward manner. By examining Figure 8, it is clear that the exponent of a horizontal edge l levels from the bottom is just $n-2-2l$, while the exponent of a vertical edge l levels above the lowest level of vertical edges is $2l+1$. There are $n-1$ edges in the bottom level of horizontal edges, and $n-1-2l$ edges l levels up. There are $n-2-2l$ vertical edges l levels up.

Therefore, if n is even,

$$\sum_{e \in E} \mathrm{ex}(e) = \sum_{l=0}^{n/2-1} (n-2-2l)(n-1-2l) + \sum_{l=0}^{n/2-2} (2l+1)(n-2-2l) =$$

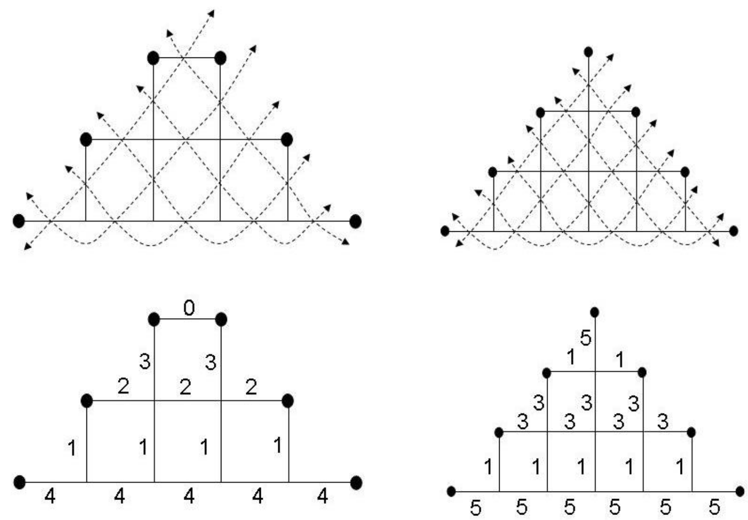

FIGURE 8. The exponents of the edges in some Tower of Hanoi graphs, which can be seen to follow regular and predictable patterns.

$$\sum_{l=0}^{n/2-1}\left(4l^2-(4n-6)l+(n^2-3n+2)\right)+\sum_{l=0}^{n/2-2}\left(-4l^2+(2n-6)l+(n-2)\right)=$$
$$\frac{2(n/2-1)(n/2)(n-1)}{3}-(4n-6)\frac{(n/2-1)(n/2)}{2}+(n^2-3n+2)(n/2)-$$
$$\frac{2(n/2-2)(n/2-1)(n-3)}{3}+(2n-6)\frac{(n/2-2)(n/2-1)}{2}+(n-2)(n/2-1)=$$
$$\frac{1}{4}n^3-\frac{1}{2}n^2.$$

Therefore,

$$\frac{1}{n}\sum_{e\in E}\mathrm{ex}(e)=\frac{1}{4}n^2-\frac{1}{2}n=|V|-n,$$

where $|V|=n/2(n/2+1)$ is the number of vertices in Γ. Now the size of a tree diagram is just $|V|-n$ because a tree diagram is a forest with $|V|$ vertices and n connected components.

Likewise, if n is odd, then

$$\sum_{e\in E}\mathrm{ex}(e)=\sum_{l=0}^{\frac{n-3}{2}}(n-2-2l)(n-1-2l)+\sum_{l=0}^{\frac{n-3}{2}}(2l+1)(n-2-2l)=$$

$$\sum_{l=0}^{\frac{n-3}{2}}\left(4l^2-(4n-6)l+(n^2-3n+2)\right)+\sum_{l=0}^{\frac{n-3}{2}}\left(-4l^2+(2n-6)l+(n-2)\right)=$$

$$\sum_{l=0}^{\frac{n-3}{2}}\left(-2nl+(n^2-2n)\right)=-n\frac{n-3}{2}\left(\frac{n-3}{2}+1\right)+(n^2-2n)\frac{n-1}{2}=$$

$$\frac{1}{4}n^3 - \frac{1}{2}n^2 + \frac{1}{4}n.$$

Therefore,

$$\frac{1}{n}\sum_{e\in E} \mathrm{ex}(e) = \frac{1}{4}n^2 - \frac{1}{2}n + \frac{1}{4} = |V| - n,$$

where $|V| = \left(\frac{n+1}{2}\right)^2$ is the number of vertices in Γ. As before, $|V|-n$ is the number of edges in a tree diagram and we are done. $\square$

Theorem 4.6. *For Γ the towers of hanoi graph with n vertices,*

$$J = \alpha\frac{\prod_{e\in E}\gamma_e^{\mathrm{ex}(e)}}{\Sigma^n}$$

for some nonzero constant α.

Proof. We already know that J is a ratio of two polynomials in the conductivities, $J = N/D$. Also, we know that $\prod_{e\in E}\gamma_e^{\mathrm{ex}(e)}\big|N$ and $D\big|\Sigma^n$.

Now J is the determinant of a matrix of partial derivatives. Each partial derivative is the derivative of a conductance with respect to a conductance, and is therefore dimensionless. J itself is then dimensionless. It follows that if we scale all the conductivities by a constant, J must remain the same. This is only possible if N and D are homogeneous polynomials of the same degree.

Now $\prod_{e\in E}\gamma_e^{\mathrm{ex}(e)}$ is already a homogeneous polynomial of degree $\sum_{e\in E}\mathrm{ex}(e)$. Likewise, Σ^n is already a homogeneous polynomial, of the same degree (by Lemma 4.5). Therefore, the only way N and D can have the same degree is if they differ from $\prod_{e\in E}\gamma_e^{\mathrm{ex}(e)}$ and Σ^n only by multiplication by a scalar. So J has the desired form. Also, the scalar constant cannot be zero, or else J would vanish everywhere. This cannot happen, however, since we already know that for all positive values of conductivities, well connected graphs are recoverable from the response matrix Λ. $\square$

5. $Y - \Delta$ INVARIANCE

In this section, we show that all other well-connected circular planar graphs have the same property as the Towers of Hanoi graph demonstrated in Theorem 4.6. We show that this property is invariant under $Y - \Delta$ transformations, which suffices because all well-connected graphs are $Y - \Delta$ equivalent (as noted on p. 20 of [2]).

To clarify notation, Σ_Γ and J_Γ will denote the expressions Σ and J for a specific graph Γ, and E_Γ will denote the set of edges of Γ.

In the following lemmas, we will consider two graphs Γ and Γ' which differ only in that Γ has a Δ between vertices a, b, and c, while Γ' has a Y at the same vertices, with center f.

Lemma 5.1.

$$\Sigma_{\Gamma'} = (\gamma_{af} + \gamma_{bf} + \gamma_{cf})\Sigma_\Gamma$$

where we are interpreting Σ_Γ as a function of the conductivities on Γ' via the substitutions

$$\gamma_{ab} = \frac{\gamma_{af}\gamma_{bf}}{\gamma_{af} + \gamma_{bf} + \gamma_{cf}}$$

$$\gamma_{ac} = \frac{\gamma_{af}\gamma_{cf}}{\gamma_{af} + \gamma_{bf} + \gamma_{cf}}$$

$$\gamma_{bc} = \frac{\gamma_{bf}\gamma_{cf}}{\gamma_{af} + \gamma_{bf} + \gamma_{cf}}$$

Proof. Identify all the boundary nodes of Γ. Then a tree diagram in Γ is just a spanning tree. Do the same for Γ'. Let Γ_0 be the common part of Γ and Γ', that is, Γ with the Δ removed or Γ' with the Y removed. Let E_0 be the set of edges of Γ_0. Define the following families of sets of edges:

$$\mathcal{I} = \{I \subseteq E_0 \ : \ I \text{ contains no cycles}\}$$

(note that I will contain a cycle if it connects any boundary nodes, because the boundary nodes have been identified)

$$\begin{aligned}
\mathcal{I}_{\{a\},\{b\},\{c\}} &= \{I \in \mathcal{I} \ : \ I \text{ does not connect any of } a,\ b, \text{ and } c\} \\
\mathcal{I}_{\{a,b\},\{c\}} &= \{I \in \mathcal{I} \ : \ I \text{ connects } a \text{ to } b \text{ but does not connect } a \text{ or } b \text{ to } c\} \\
\mathcal{I}_{\{a\},\{b,c\}} &= \{I \in \mathcal{I} \ : \ I \text{ connects } b \text{ to } c \text{ but does not connect } a \text{ to } b \text{ or } c\} \\
\mathcal{I}_{\{a,c\},\{b\}} &= \{I \in \mathcal{I} \ : \ I \text{ connects } a \text{ to } c \text{ but does not connect } a \text{ or } c \text{ to } b\} \\
\mathcal{I}_{\{a,b,c\}} &= \{I \in \mathcal{I} \ : \ I \text{ connects } a,\ b, \text{ and } c\}
\end{aligned}$$

(some of these may be empty; for example, if a and b are both boundary nodes, in which case all sets will connect them...)

$$\begin{aligned}
\mathcal{T}_{\{a\},\{b\},\{c\}} &= \{T \in \mathcal{I}_{\{a\},\{b\},\{c\}} \ : \ \text{No strict superset of } T \text{ is in } \mathcal{I}_{\{a\},\{b\},\{c\}}\} \\
\mathcal{T}_{\{a,b\},\{c\}} &= \{T \in \mathcal{I}_{\{a,b\},\{c\}} \ : \ \text{No strict superset of } T \text{ is in } \mathcal{I}_{\{a,b\},\{c\}}\} \\
\mathcal{T}_{\{a\},\{b,c\}} &= \{T \in \mathcal{I}_{\{a\},\{b,c\}} \ : \ \text{No strict superset of } T \text{ is in } \mathcal{I}_{\{a\},\{b,c\}}\} \\
\mathcal{T}_{\{a,c\},\{b\}} &= \{T \in \mathcal{I}_{\{a,c\},\{b\}} \ : \ \text{No strict superset of } T \text{ is in } \mathcal{I}_{\{a,c\},\{b\}}\} \\
\mathcal{T}_{\{a,b,c\}} &= \{T \in \mathcal{I}_{\{a,b,c\}} \ : \ \text{No strict superset of } T \text{ is in } \mathcal{I}_{\{a,b,c\}}\}
\end{aligned}$$

Then define the polynomials

$$\begin{aligned}
\Sigma_{\{a\},\{b\},\{c\}} &= \sum_{T \in \mathcal{T}_{\{a\},\{b\},\{c\}}} \prod_{e \in T} \gamma_e \\
\Sigma_{\{a,b\},\{c\}} &= \sum_{T \in \mathcal{T}_{\{a,b\},\{c\}}} \prod_{e \in T} \gamma_e \\
\Sigma_{\{a\},\{b,c\}} &= \sum_{T \in \mathcal{T}_{\{a\},\{b,c\}}} \prod_{e \in T} \gamma_e \\
\Sigma_{\{a,c\},\{b\}} &= \sum_{T \in \mathcal{T}_{\{a,c\},\{b\}}} \prod_{e \in T} \gamma_e \\
\Sigma_{\{a,b,c\}} &= \sum_{T \in \mathcal{T}_{\{a,b,c\}}} \prod_{e \in T} \gamma_e
\end{aligned}$$

By considering the terms involved, it is now straightforward to check that

$$\Sigma_\Gamma = (\gamma_{ab}\gamma_{ac} + \gamma_{ab}\gamma_{bc} + \gamma_{ac}\gamma_{bc})\Sigma_{\{a\},\{b\},\{c\}} + (\gamma_{ac} + \gamma_{bc})\Sigma_{\{a,b\},\{c\}} + \\ (\gamma_{ab} + \gamma_{ac})\Sigma_{\{a\},\{b,c\}} + (\gamma_{ab} + \gamma_{bc})\Sigma_{\{a,c\},\{b\}} + \Sigma_{\{a,b,c\}}$$

while

$$\Sigma_{\Gamma'} = \gamma_{af}\gamma_{bf}\gamma_{cf}\Sigma_{\{a\},\{b\},\{c\}} + (\gamma_{af} + \gamma_{bf})\gamma_{cf}\Sigma_{\{a,b\},\{c\}} + \\ (\gamma_{bf} + \gamma_{cf})\gamma_{af}\Sigma_{\{a\},\{b,c\}} + (\gamma_{af} + \gamma_{cf})\gamma_{bf}\Sigma_{\{a,c\},\{b\}} + (\gamma_{af} + \gamma_{bf} + \gamma_{cf})\Sigma_{\{a,b,c\}}$$

Then we are done, because, with the substitutions specified,

$$\gamma_{ab}\gamma_{ac} + \gamma_{ab}\gamma_{bc} + \gamma_{ac}\gamma_{bc} = \frac{\gamma_{af}\gamma_{bf}\gamma_{cf}}{\gamma_{af} + \gamma_{bf} + \gamma_{cf}}$$

$$\gamma_{ac} + \gamma_{bc} = \frac{\gamma_{af}\gamma_{cf} + \gamma_{bf}\gamma_{cf}}{\gamma_{af} + \gamma_{bf} + \gamma_{cf}}$$

(and so on), and

$$1 = \frac{\gamma_{af} + \gamma_{bf} + \gamma_{cf}}{\gamma_{af} + \gamma_{bf} + \gamma_{cf}}$$

□

Lemma 5.2.

$$J_{\Gamma'} = \frac{\pm\gamma_{af}\gamma_{bf}\gamma_{cf}}{(\gamma_{af} + \gamma_{bf} + \gamma_{cf})^3} J_\Gamma$$

where we are interpreting J_Γ as a function of the conductivities on Γ' via the substitutions

$$\gamma_{ab} = \frac{\gamma_{af}\gamma_{bf}}{\gamma_{af} + \gamma_{bf} + \gamma_{cf}}$$
$$\gamma_{ac} = \frac{\gamma_{af}\gamma_{cf}}{\gamma_{af} + \gamma_{bf} + \gamma_{cf}}$$
$$\gamma_{bc} = \frac{\gamma_{bf}\gamma_{cf}}{\gamma_{af} + \gamma_{bf} + \gamma_{cf}}$$

Proof. The response map of Γ' can be computed from the conductivities in two steps: first perform a $Y - \Delta$ transformation to produce conductivities on Γ, and then use the map from conductivities to response matrix for Γ. Then, by the chain rule, we have

$$J_{\Gamma'} = \frac{\partial(\gamma_{ab}, \gamma_{ac}, \gamma_{bc})}{\partial(\gamma_{af}, \gamma_{bf}, \gamma_{cf})} J_\Gamma.$$

The Jacobian determinant here can be computed directly. After a long computation, it turns out that

$$\frac{\partial(\gamma_{ab}, \gamma_{ac}, \gamma_{bc})}{\partial(\gamma_{af}, \gamma_{bf}, \gamma_{cf})} = \frac{\pm\gamma_{af}\gamma_{bf}\gamma_{cf}}{(\gamma_{af} + \gamma_{bf} + \gamma_{cf})^3}$$

and we are done □

Lemma 5.3. *If n is the number of boundary nodes in Γ (which is the same as in Γ'), then*

$$\mathrm{ex}(af) = n - 2 - \mathrm{ex}(bc)$$
$$\mathrm{ex}(bf) = n - 2 - \mathrm{ex}(ac)$$
$$\mathrm{ex}(cf) = n - 2 - \mathrm{ex}(ab)$$

Proof. This is obvious from Figure 9. The geodesics of Γ and Γ' can be identified in an obvious way, and the geodesics which count towards $\mathrm{ex}(af)$ are exactly those which do not count towards $\mathrm{ex}(bc)$ (not counting the two geodesics which cross at (a, f) or at (b, c)). □

Theorem 5.4. *If Γ and Γ' are well-connected circular planar graphs with n vertices, related by a $Y - \Delta$ transformation, and if*

$$J_\Gamma = \frac{\alpha \prod_{e \in E_\Gamma} \gamma_e^{\mathrm{ex}(e)}}{\Sigma_\Gamma^n}$$

then

$$J_{\Gamma'} = \frac{\pm\alpha \prod_{e \in E_{\Gamma'}} \gamma_e^{\mathrm{ex}(e)}}{\Sigma_{\Gamma'}^n}$$

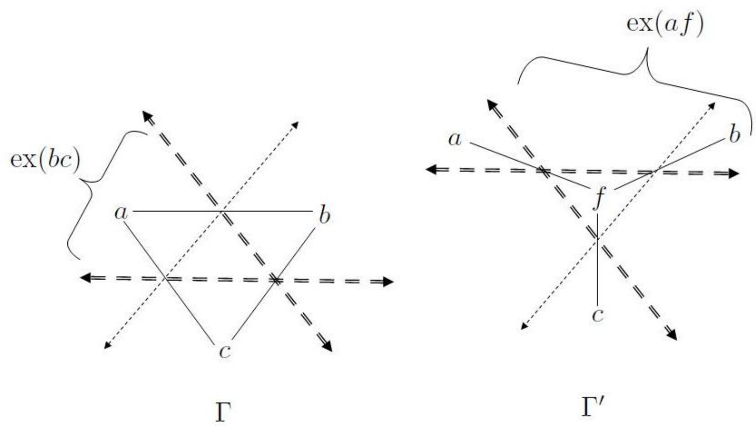

FIGURE 9. Lemma 5.3: the values ex(af) and ex(bc) just come from the number of geodesics between the two bolded geodesics. Every geodesic except for these two will count for exactly one of ex(af) or ex(bc), and there are n geodesics total.

Proof. Suppose first that Γ has a Δ and Γ' has a Y at the vertices a, b, and c, and $f \in \Gamma'$ is the center of the Y. Then from Lemmas 5.1 and 5.2,

$$J_{\Gamma'} = \frac{\pm\gamma_{af}\gamma_{bf}\gamma_{cf}}{(\gamma_{af}+\gamma_{bf}+\gamma_{cf})^3} J_\Gamma = \frac{\pm\alpha\gamma_{af}\gamma_{bf}\gamma_{cf}\prod_{e\in E_\Gamma}\gamma_e^{\mathrm{ex}(e)}}{(\gamma_{af}+\gamma_{bf}+\gamma_{cf})^3\,\Sigma_\Gamma^n} =$$
$$\frac{\pm\alpha\gamma_{af}\gamma_{bf}\gamma_{cf}(\gamma_{af}+\gamma_{bf}+\gamma_{cf})^{n-3}\prod_{e\in E_\Gamma}\gamma_e^{\mathrm{ex}(e)}}{\Sigma_{\Gamma'}^n}$$

where as usual we interpret the conductivities of Γ as functions of the conductivities of Γ' via the usual substitutions. Now clearly,

$$\prod_{e\in E_\Gamma}\gamma_e^{\mathrm{ex}(e)} = \frac{\gamma_{ab}^{\mathrm{ex}(ab)}\gamma_{ac}^{\mathrm{ex}(ac)}\gamma_{bc}^{\mathrm{ex}(bc)}}{\gamma_{af}^{\mathrm{ex}(af)}\gamma_{bf}^{\mathrm{ex}(bf)}\gamma_{cf}^{\mathrm{ex}(cf)}}\prod_{e\in E_{\Gamma'}}\gamma_e^{\mathrm{ex}(e)} =$$
$$\frac{\left(\frac{\gamma_{af}\gamma_{bf}}{\gamma_{af}+\gamma_{bf}+\gamma_{cf}}\right)^{n-2-\mathrm{ex}(cf)}\left(\frac{\gamma_{af}\gamma_{cf}}{\gamma_{af}+\gamma_{bf}+\gamma_{cf}}\right)^{n-2-\mathrm{ex}(bf)}\left(\frac{\gamma_{bf}\gamma_{cf}}{\gamma_{af}+\gamma_{bf}+\gamma_{cf}}\right)^{n-2-\mathrm{ex}(af)}}{\gamma_{af}^{\mathrm{ex}(af)}\gamma_{bf}^{\mathrm{ex}(bf)}\gamma_{cf}^{\mathrm{ex}(cf)}}\prod_{e\in E_{\Gamma'}}\gamma_e^{\mathrm{ex}(e)} =$$
$$\frac{(\gamma_{af}\gamma_{bf}\gamma_{cf})^{2n-4-\mathrm{ex}(af)-\mathrm{ex}(bf)-\mathrm{ex}(cf)}}{(\gamma_{af}+\gamma_{bf}+\gamma_{cf})^{3n-6-\mathrm{ex}(af)-\mathrm{ex}(bf)-\mathrm{ex}(cf)}}\prod_{e\in E_{\Gamma'}}\gamma_e^{\mathrm{ex}(e)}$$

using Lemma 5.3. From Figure 10 is is straightforward to check that $\mathrm{ex}(af)+\mathrm{ex}(bf)+\mathrm{ex}(cf)=2n-3$, so

$$\prod_{e\in E_\Gamma}\gamma_e^{\mathrm{ex}(e)} = \frac{(\gamma_{af}\gamma_{bf}\gamma_{cf})^{-1}}{(\gamma_{af}+\gamma_{bf}+\gamma_{cf})^{n-3}}\prod_{e\in E_{\Gamma'}}\gamma_e^{\mathrm{ex}(e)}$$

Then

$$J_{\Gamma'} = \frac{\pm\alpha\gamma_{af}\gamma_{bf}\gamma_{cf}(\gamma_{af}+\gamma_{bf}+\gamma_{cf})^{n-3}\prod_{e\in E_\Gamma}\gamma_e^{\mathrm{ex}(e)}}{\Sigma_{\Gamma'}^n} =$$

$$\frac{\pm\alpha\gamma_{af}\gamma_{bf}\gamma_{cf}\left(\gamma_{af}+\gamma_{bf}+\gamma_{cf}\right)^{n-3}\left(\gamma_{af}\gamma_{bf}\gamma_{cf}\right)^{-1}\prod_{e\in E_{\Gamma'}}\gamma_e^{\mathrm{ex}(e)}}{\left(\gamma_{af}+\gamma_{bf}+\gamma_{cf}\right)^{n-3}\Sigma^n_{\Gamma'}} = \frac{\pm\alpha\prod_{e\in E_{\Gamma'}}\gamma_e^{\mathrm{ex}(e)}}{\Sigma^n_{\Gamma'}}$$

The other case, where Γ has the Y and Γ' the Δ, is handled similarly. □

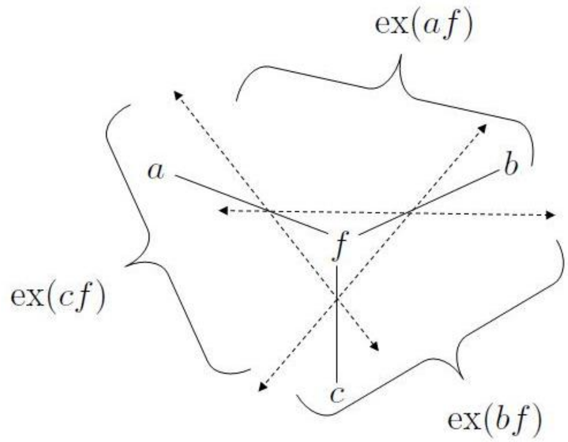

FIGURE 10. Each geodesic endpoint except for three counts towards exactly one of ex(af), ex(bf), and ex(cf). There are n geodesics, and $2n$ endpoints, so the sum of these three numbers is $2n-3$.

Now, since all well-connected graphs on n boundary vertices are $Y-\Delta$ equivalent (because of Theorem 8.7 in [2]), we have the following:

Theorem 5.5. *If Γ is a well connected graph with n boundary vertices, then*

$$J_\Gamma = \frac{\pm\alpha_n\prod_{e\in E_\Gamma}\gamma_e^{\mathrm{ex}(e)}}{\Sigma^n_\Gamma}$$

where α_n is a nonzero constant depending only on n.

6. Recovery of Negative Conductances

All of the preceding has been under the tacit assumption that all conductivities are positive. (This ensures that Σ and Υ do not vanish, so that Λ and H are well defined.) However, if we let some of the γ_i be negative or zero or complex, Σ may still be nonzero, and if this is the case, then Λ will be well-defined, because the Schur complement formula is still valid. Moreover, the formula for the Jacobian determinant will still hold for most of these values.

Let L be the map from conductivities γ to the response map Λ. L is a rational map from some dense subset of $\mathbb{R}^{|E|}$ to Kir_n. In fact, the domain of L is precisely the γ for which Σ does not vanish. If the network in question is well-connected, then the recovery algorithm of [2] establishes that the map L is in fact a birational map. That is, there is some rational map R from some dense subset of Kir_n to $\mathbb{R}^{|E|}$ such that $R\circ L$ and $L\circ R$ are identity functions generically.

Lemma 6.1. *If $\gamma \in \mathbb{R}^{|E|}$ is a collection of conductivities, all nonzero, and $L(\gamma)$ exists, then*

$$\gamma = \lim_{\Lambda \to L(\gamma)} R(\Lambda),$$

where Λ ranges over the domain of R. In particular, the limit is well-defined and exists.

Proof. Since the domain of R is dense, there are certainly Λ in the domain of R arbitrarily close to $L(\gamma)$. To show that the limit exists, we show that R can be extended to a continuous function on a neighborhood of $L(\gamma)$. From Theorem 5.5 and the inverse function theorem, we know that we can find a local inverse of $L(\gamma)$ about γ, since $L(\gamma)$ exists and the Jacobian determinant of L is nonzero. This gives a continuous function R' which must agree with R on the intersection of their domains. Therefore, $\lim_{\Lambda \to L(\gamma)} R(\Lambda)$ exists and equals $R'(L(\gamma)) = \gamma$. □

We now lift the restriction that Γ be well-connected.

Theorem 6.2. *If Γ is a critical circular planar graph, and γ is a vector of conductivities on the edges of Γ, and the entries in γ are all nonzero (but possibly negative), and if Λ exists for these values of γ, then γ is recoverable from Λ. In other words, if we have another γ' satisfying the same constraints of γ', then its response matrix Λ' must differ from Λ.*

Proof. If Γ is well connected, this follows directly from Lemma 6.1. Otherwise, take Γ and add boundary spikes and boundary-to-boundary edges along the boundary until Γ becomes well connected. This is known to be possible. Assign the new edges random nonzero conductivities. With probability 1, the resulting network will not have vanishing Σ. This can easily be seen at each step. To wit, adding a boundary-to-boundary edge does not effect Σ at all, and the effect of adding a boundary spike with conductance c is to turn Σ into $\Sigma' = c\Sigma + \Phi$ for some other polynomial Φ. Then by choosing a random c, Σ' will not vanish.

Once the new graph is constructed, we use the fact that well-connected critical circular planar graphs can be recovered with negative conductivities, and we are done. □

This theorem was possibly proven by Michael Goff in an earlier paper, [4], via completely different means, but his proof has certain flaws, and it is not clear whether it can be salvaged. A third, completely different proof of this result was also found by the present author, in the context of nonlinear networks.

7. Future Work

Theorem 5.5 was originally a conjecture, but the conjecture had a slight difference: there was no α_n. By doing symbolic calculations using SAGE, I showed that for n up to 6, $\alpha_n = \pm 1$. It seems very likely that this continues to be true for all n. However, I don't see how to prove this.

In many ways, such a simple result as Theorem 5.5 ought to have a simpler proof. The current proof uses many unlikely algebraic results (such as the implicit use of the fact that polynomial rings are unique factorization domains), as well as tedious verifications (such as those in §4) and pictures.

The extent to which §4 is tied to the Towers of Hanoi graphs is troubling. It seems like there should be a more general way to determine the amount of degeneracy of a graph. It is not even obvious without thinking about electrical circuits that multiple ways of reducing a medial graph should yield the same medial graph at the end. Also there is no clear combinatorial way of understanding the amount of degeneracy – for example, counting lenses in the medial graph does not work (Figure 11). It seems possible that the formula in Theorem 5.5 says something about how degenerate a graph is, if it is obtained by deleting one edge from a well-connected critical graph.

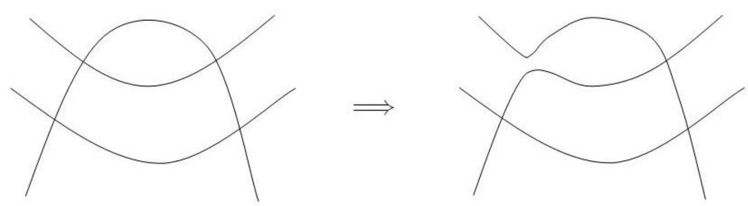

FIGURE 11. In the diagram on the left, there are two lenses. However, there is only one degree of degeneracy, since after removing the simple lens, no lenses remain. Statically counting lenses is not the correct way to measure degeneracy.

Also, the simple formula in Theorem 5.5 suggests that the algebraic geometry of the map from γ to Λ might be worth studying. For example, it might turn out that networks could be classified by the "degree" of the map from γ to Λ in some sense of the word, and then perhaps edge deletions and contractions do not increase degree.

The original motivation for this work is the idea of applying it to a generalized version of $Y-\Delta$ transformations. A $Y-\Delta$ transformation can be seen as a switch between a complete graph and a well-connected circular planar graph with the same number of vertices. In general, we can switch between a K_n and a well-connected critical circular planar graph with n boundary vertices. In one direction this correspondence is straightforward, but in the other, negative and zero conductivities can be introduced. These sort of transformations might be useful for turning nonplanar graphs into planar ones. This application of Theorem 6.2 was noted by Michael Goff.

For example, it can be shown using Theorem 6.2 and some additional algebra that the graph pictured in Figure 12 is recoverable, but lattice graphs with two or more K_4s inserted are not.

8. References

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[3] Giansiracusa, Jeffrey, "The Map L and Partial Recovery in Circular Planar Non-Critical Networks", 2003.

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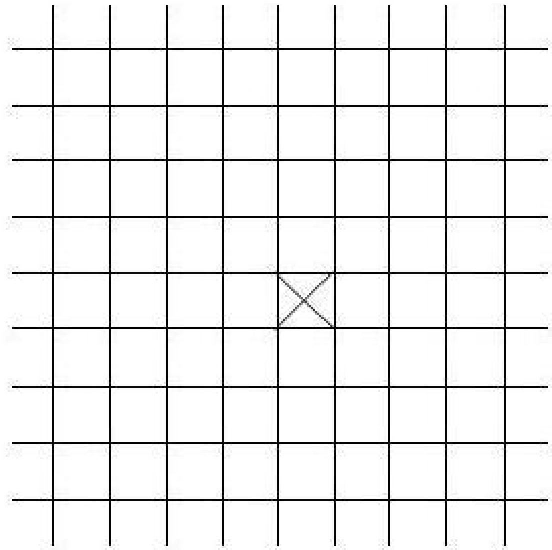

FIGURE 12. A lattice graph in which one square has been replaced by a complete graph on four vertices, K_4. With boundary nodes along the outside, this graph is recoverable. This can be shown by replacing the K_4 with a well-connected planar graph with 4 boundary nodes. There are some tricky cases where some of the conductances on the well-connected planar graph vanish, but they can be handled. On the other hand, if two or more K_4's are inserted, the graph becomes nonrecoverable (though it is generically recoverable).

[5] Lewandowski, Matthew J., "Determinant of a Principle Proper Submatrix of the Kirchhoff Matrix", 2008.
[6] Perry, Karen, "Discrete Complex Analysis", 2003.